HEAT TRANSFER IN A CYLINDRICAL UNDERGROUND CHANNEL

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We treat the problem of the flow of a fluid in a cylindrical underground channel, taking account of the thermal interaction of the stream with the surrounding medium.

The problem of fluid flow, taking account of heat transfer, arises in the study of heat transfer between the ground and various kinds of piping such as gas and oil pipelines, steam and water mains, etc.

We consider a cylindrical channel of radius R_0 with its axis at a depth H below the surface of the ground.

We make the following assumptions:

1) the ground is isotropic, i.e., its properties are the same in all directions;

2) the distance from the surface of the ground to the channel axis is the same over the whole length of the channel, i.e., H = idem;

3) the temperature distribution in the ground far from the channel ($x \to \infty$) is described by the equation $T_{gr} = T_e + \delta y$, where $\delta = (T_{nl} = T_e)/H_0$.

For steady-state heat transfer between the channel and the ground the temperature distribution of the ground is given by Laplace's equation

$$\frac{\partial^2 T_{gr}}{\partial x^2} + \frac{\partial^2 T_{gr}}{\partial y^2} = 0. \tag{1}$$

We seek the solution of the problem in the form

$$T_{gr} = T_e + \delta y + T_2, \tag{2}$$

where T_2 satisfies Laplace's equation

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0. \tag{3}$$

We consider the solution of the problem for boundary conditions of the first and third kinds.

Boundary conditions of the first kind have the form

$$r = R_0, \quad T_{gr} = T_F, \tag{4a}$$

$$y = 0, \quad T_{gr} = T_e. \tag{4b}$$

Taking account of (2) and (3) the boundary conditions for T_2 are

$$r = R_0, \quad T_2 = T_F - T_e - \delta y|_{r=R_0}, \tag{5a}$$

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$$y = 0, \quad T_2 = 0. \tag{5b}$$

We seek the solution of (3) in bipolar coordinates [1, 2]. To do this we use the modified system of bipolar coordinates

$$x + iy = ai \operatorname{cth} \frac{\alpha + i\beta}{2}, \tag{6}$$

where

$$x = \frac{a \sin\beta}{\operatorname{ch}\alpha - \cos\beta}; \quad y = \frac{a \operatorname{sh}\alpha}{\operatorname{ch}\alpha - \cos\beta}.$$

Since the representation (6) is conformal, Laplace's equation (3) is invariant, i.e.,

$$\nabla T_2 = \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = g^2 \left(\frac{\partial^2 T_2}{\partial \alpha^2} + \frac{\partial^2 T_2}{\partial \beta^2} \right) = 0, \tag{7}$$

where $g = (\operatorname{ch}\alpha - \cos\beta)a^{-1}$ is the scale of the transformation; $a = \sqrt{H^2 - R_0^2}$.

Instead of (7) we can write

$$\frac{\partial^2 T_2}{\partial \alpha^2} + \frac{\partial^2 T_2}{\partial \beta^2} = 0 \tag{8}$$

with the boundary conditions

$$\alpha = \alpha_0, \quad T_2 = T_F - T_e + \frac{\delta a \operatorname{sh}\alpha_0}{\operatorname{ch}\alpha_0 - \cos\beta}; \tag{9a}$$

$$\alpha = 0, \quad T_2 = 0. \tag{9b}$$

Since the region for which the temperature distribution of the ground is being constructed is symmetrical with respect to the $\beta(0, y)$ axis, the function describing the temperature distribution of the ground must be an even function of $\beta(x)$. Therefore we seek the solution in the form

$$T_2 = A + B\alpha + \sum_{n=1}^{\infty} (A_n \operatorname{ch} n\alpha + B_n \operatorname{sh} n\alpha) \cos n\beta. \tag{10}$$

By satisfying the boundary conditions, determining the constants A, B, A_n, and B_n, and substituting them into (10) we obtain the solution of the problem for boundary conditions of the first kind

$$T_{gr} = T_e + \delta a \frac{\operatorname{sh}\alpha}{\operatorname{ch}\alpha - \cos\beta} + (T_F - T_e - \delta a) \frac{\alpha}{\alpha_0} - 2\delta a \sum_{n=1}^{\infty} \frac{\operatorname{sh} n\alpha}{\operatorname{sh} n\alpha_0} \exp(-n\alpha_0) \cos n\beta. \tag{11}$$

The boundary conditions of the third kind have the form

$$r = R_0, \quad \frac{\partial T_{gr}}{\partial n} = -\frac{\mathrm{Bi}_1}{R_0} (T_F - T_{gr}), \tag{12a}$$

$$y = 0, \quad \frac{\partial T_{gr}}{\partial y} = \frac{\mathrm{Bi}_2}{H} (T_{gr} - T_e). \tag{12b}$$

We rewrite conditions (12a) and (12b) in bipolar coordinates: for $\alpha = \alpha_0$

$$\frac{\operatorname{ch}\alpha_0 - \cos\beta}{a} \cdot \frac{\partial T_2}{\partial \alpha} = (T_F - T_e - T_2) \frac{\mathrm{Bi}_1}{R_0} - \delta a \frac{\mathrm{Bi}_1}{R_0} \cdot \frac{\operatorname{ch}\alpha_0}{\operatorname{ch}\alpha_0 - \cos\beta} - \delta \frac{1 - \operatorname{ch}\alpha_0 \cos\beta}{\operatorname{ch}\alpha_0 - \cos\beta}; \tag{13a}$$

for $\alpha = 0$

$$\frac{1-\cos\beta}{a}\cdot\frac{\partial T_2}{\partial\alpha}=\mathrm{Bi}_2\frac{T_e}{H}-\delta. \tag{13b}$$

The coefficients in the boundary conditions (13a,b) depend on the parameter β. The solution will be in the form of nonorthogonal series, which complicates the determination of the coefficients. We simplify the problem by linearizing $\tilde{g}(\alpha,\beta) = (\mathrm{ch}\,\alpha - \cos\beta)/a$. We average in such a way that

$$F(\tilde{g})=\int_0^{\alpha_0}\int_{-\pi}^{\pi}[\bar{g}(\alpha,\ \beta)-\tilde{g}]^2\,d\alpha d\beta=\min,$$

$$\frac{dF}{d\tilde{g}}=2\int_0^{\alpha_0}\int_{-\pi}^{\pi}[\bar{g}(\alpha,\ \beta)-\tilde{g}]\,d\alpha d\beta=0\ \left(\frac{d^2F}{d\tilde{g}^2}>0\right).$$

Hence

$$\tilde{g}=\frac{1}{2\pi\alpha_0}\int_0^{\alpha_0}\int_{-\pi}^{\pi}\bar{g}(\alpha,\ \beta)\,d\alpha d\beta. \tag{14}$$

We rewrite boundary conditions (13a,b), taking account of the linearization performed

$$\alpha=\alpha_0,\quad \frac{\partial T_2}{\partial\alpha}=\frac{\mathrm{Bi}_1'}{\mathrm{ch}\,\alpha_0}\left(T_F-T_e-\delta a\frac{\mathrm{sh}\,\alpha_0}{\mathrm{ch}\,\alpha_0-\cos\beta}-T_2\right)-\delta a\frac{1-\mathrm{ch}\,\alpha_0\cos\beta}{(\mathrm{ch}\,\alpha_0-\cos\beta)^2}, \tag{15a}$$

$$\alpha=0,\quad \frac{\partial T_2}{\partial\alpha}=\mathrm{Bi}_2'T_2-\delta a, \tag{15b}$$

where $\mathrm{Bi}_1' = \alpha_1 a/\lambda_{gr}$ and $\mathrm{Bi}_2' = \alpha_e a/\lambda_{gr}$.

Using (15a) and (15b) the solution of the problem for boundary conditions of the third kind takes the form

$$T_{gr}=T_e+\delta a\frac{\mathrm{sh}\,\alpha}{\mathrm{ch}\,\alpha-\cos\beta}+\frac{\mathrm{Bi}_1'\left[\delta a\left(\alpha_0-\alpha+\frac{\mathrm{ch}\,\alpha}{\mathrm{Bi}_1'}\right)+(T_F-T_e-\delta a)(1+\alpha\,\mathrm{Bi}_2')\right]}{\mathrm{Bi}_1'(1+\alpha_0\,\mathrm{Bi}_2')+\mathrm{Bi}_2'\,\mathrm{ch}\,\alpha_0}-$$

$$-2\delta a\sum_{n=1}^{\infty}\frac{(\mathrm{Bi}_1'-n\,\mathrm{ch}\,\alpha_0)\left(\frac{n}{\mathrm{Bi}_2'}\mathrm{ch}\,n\alpha+\mathrm{sh}\,n\alpha\right)\exp(-n\alpha_0)\cos n\beta}{\mathrm{Bi}_1'\left(\frac{n}{\mathrm{Bi}_2'}\mathrm{ch}\,n\alpha_0+\mathrm{sh}\,n\alpha_0\right)+n\,\mathrm{ch}\,\alpha_0\left(\frac{n}{\mathrm{Bi}_2'}\mathrm{sh}\,n\alpha_0+\mathrm{ch}\,n\alpha_0\right)}. \tag{16}$$

We consider the limiting cases:

a) $\mathrm{Bi}_1' \to \infty$, $\mathrm{Bi}_2' \to \infty$

$$\lim_{\substack{\mathrm{Bi}_1'\to\infty\\ \mathrm{Bi}_2'\to\infty}} T_{gr}=T_e+\delta a\frac{\mathrm{sh}\,\alpha}{\mathrm{ch}\,\alpha-\cos\beta}+(T_F-T_e-\delta a)\frac{\alpha}{\alpha_0}-2\delta a\sum_{n=1}^{\infty}\frac{\mathrm{sh}\,n\alpha}{\mathrm{sh}\,n\alpha_0}\exp(-n\alpha_0)\cos n\beta.$$

In this case the boundary conditions of the third kind degenerate into conditions of the first kind, and solution (16) goes over into (11).

It is easy to see that $\mathrm{Bi}_1' \to \infty$ and $\mathrm{Bi}_2' \to \infty$ can occur if

1) $\alpha_1 \to \infty$, $\alpha_2 \to \infty$; 2) $\lambda_{gr} \to 0$.

The first case corresponds to intense heat transfer between the fluid and the channel wall and between the external medium and the surface of the ground ($Bi_1' > 100$). The second case is characteristic of a channel in material of very low thermal conductivity, such as dry ground or thermal insulation.

b) $Bi_1' \to \infty$, $Bi_2' \to 0$

$$\lim_{\substack{Bi_1' \to \infty \\ Bi_2' \to \infty}} T_{gr} = T_e + \delta a \frac{\operatorname{sh} \alpha}{\operatorname{ch} \alpha - \cos \beta} + \delta a (\alpha_0 - \alpha - 1) - 2\delta a \sum_{n=1}^{\infty} \frac{\operatorname{ch} n\alpha}{\operatorname{ch} n\alpha_0} \exp(-n\alpha_0) \cos n\beta.$$

The conditions $Bi_1' \to \infty$ and $Bi_2' \to \infty$ can be realized if $\alpha_1 \to \infty$ and $\alpha_e \to 0$. This case corresponds to intense heat transfer between the fluid and the channel wall, and no heat transfer between the fluid and the external medium (e.g., the surface of the ground is insulated by a thick blanket of snow).

c) We estimate the value of the series in (16). For all $0 \le \alpha \le \alpha_0$ the following inequality holds:

$$F = \left| 2\delta a \sum_{n=1}^{\infty} \frac{\exp(-n\alpha_0)(Bi_1' - n \operatorname{ch} \alpha_0)}{Bi_1' + n \operatorname{ch} \alpha_0 \dfrac{\dfrac{n}{Bi_2'} \operatorname{th} n\alpha_0 + 1}{\dfrac{n}{Bi_2'} + \operatorname{th} n\alpha_0}} \; \frac{\dfrac{n}{Bi_2'} \operatorname{ch} n\alpha + \operatorname{sh} n\alpha}{\dfrac{n}{Bi_2'} \operatorname{ch} n\alpha_0 + \operatorname{sh} n\alpha_0} \cos n\beta \right| \leqslant$$

$$\leqslant 2\delta a \left| \sum_{n=1}^{\infty} \frac{\exp(-n\alpha_0)(Bi_1' - n \operatorname{ch} \alpha_0) \cos n\beta}{Bi_1' + n \operatorname{ch} \alpha_0 \dfrac{\dfrac{n}{Bi_2'} \operatorname{th} n\alpha_0 + 1}{\dfrac{n}{Bi_2'} + \operatorname{th} n\alpha_0}} \right| <$$

$$< 2\delta a \left| \sum_{n=1}^{\infty} \frac{\exp(-n\alpha_0)(Bi_1' - n \operatorname{ch} \alpha_0) \cos n\beta}{Bi_1' + n \operatorname{ch} \alpha_0} \right| < 2\delta a \left| \sum_{n=1}^{\infty} \exp(-n\alpha_0) \cos n\beta \right|. \tag{17}$$

Using the relation

$$\frac{\operatorname{sh} \alpha_0}{\operatorname{ch} \alpha_0 - \cos \beta} = 1 + 2 \sum_{n=1}^{\infty} \exp(-n\alpha_0) \cos n\beta,$$

we obtain

$$F < \delta a \left| \frac{\operatorname{sh} \alpha_0}{\operatorname{ch} \alpha_0 - \cos \beta} - 1 \right| = \delta \left| y_{\alpha=\alpha_0} - a \right|. \tag{18}$$

If we take $\delta = 1°C/m$, the sum of the series does not exceed 0.7-0.8°C. This estimate shows that within the limits of accuracy needed in engineering calculations this series can be neglected. Equation (16) then simplifies to

$$T_{gr} = T_e + \delta a \frac{\operatorname{sh} \alpha}{\operatorname{ch} \alpha - \cos \beta} + \frac{Bi_1' \left[\delta a \left(\alpha_0 - \alpha + \dfrac{\operatorname{ch} \alpha}{Bi_1'} \right) + (T_F - T_e - \delta a)(1 + \alpha Bi_2') \right]}{Bi_1'(1 + \alpha_0 Bi_2') + Bi_2' \operatorname{ch} \alpha_0}. \tag{19}$$

The equations for the isotherms of the temperature distribution of the ground around a pipe can be obtained from (19):

$$\beta = \arccos \left\{ \operatorname{ch} \alpha - \frac{\delta a \operatorname{sh} \alpha}{T_{gr} - T_e - \dfrac{Bi_1' [\delta a (\alpha_0 - \alpha + \operatorname{ch}\alpha / Bi_1') + (T_F - T_e - \delta a)(1 + \alpha Bi_2')]}{Bi_1'(1 + \alpha_0 Bi_2') + Bi_2' \operatorname{ch} \alpha_0}} \right\}. \tag{20}$$

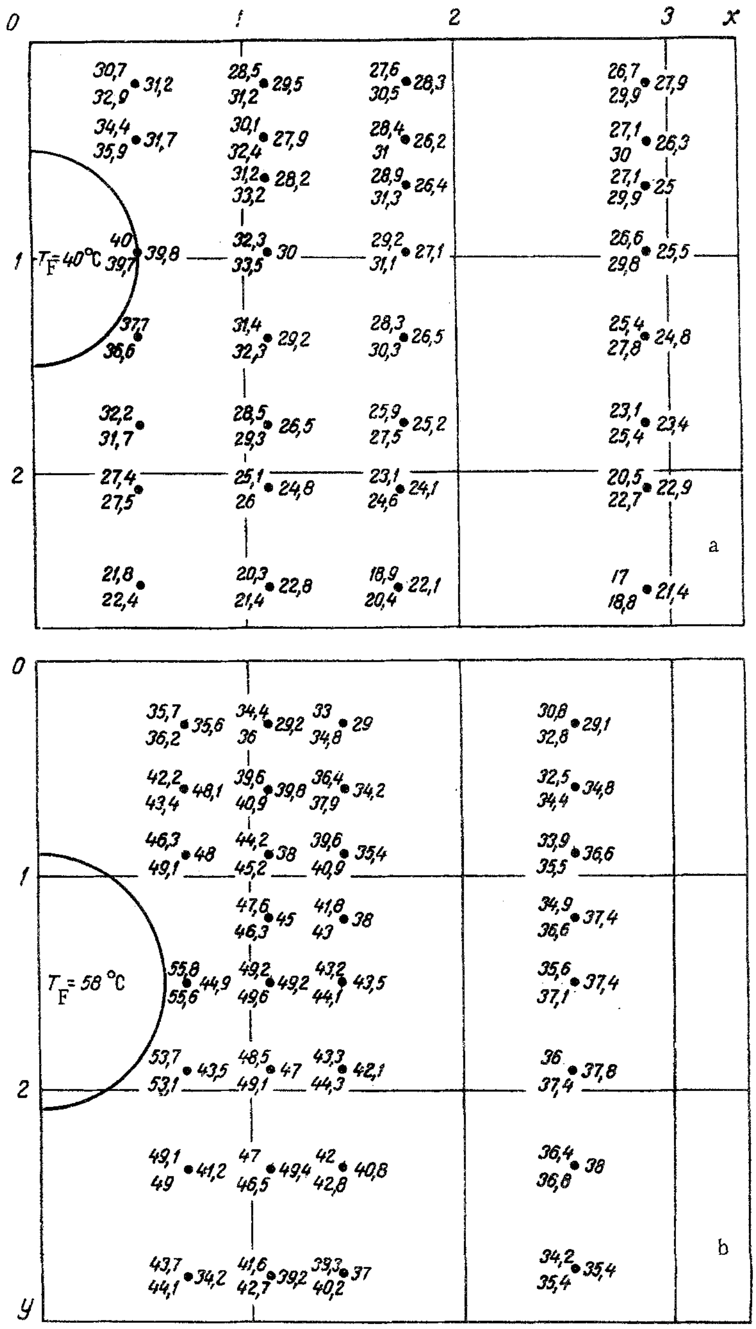

Fig. 1. Temperature distribution of the ground around a gas main at Middle Asia Center (a) and Bukhara-Ural (b) for T_e = 26.5 and 29°C, respectively; x and y are in meters.

We consider the change in the fluid temperature along the axis of the cylindrical channel, assuming that the cross flow of heat in the ground is much larger than the heat transferred by thermal conduction in the fluid, i.e.,

$$\alpha_1 (T_{gr} - T_{gr\ r=R_0}) \gg \lambda_F \frac{\partial T_F}{\partial z}.$$

In this case the heat-balance equation in bipolar coordinates for an elementary portion of the channel of length dz takes the form

$$\alpha_1 \left[\int_{-\pi}^{\pi} (T_F - T_{gr}|_{\alpha=\alpha_0})\, a \frac{R_0 \operatorname{sh} \alpha_0}{\operatorname{ch} \alpha_0 - \cos \beta} d\beta \right] dz = -Gc_p dT_F. \tag{21}$$

After substituting the value of $T_{gr}|_{\alpha=\alpha_0}$ from (19) into this equation, integrating, and using the condition that $T = T_{ng}$ at $z = 0$ we obtain

$$T_F = T_e + \frac{b_1}{c_1}\delta a + \left(T_{ng} - T_e - \frac{b_1}{c_1}\delta a\right) \exp\left(-\frac{\tilde{k}\pi Dz}{Gc_p}\right),$$

where

$$c_1 = \frac{\mathrm{Bi}_1' \mathrm{Bi}_2'}{\mathrm{Bi}_1'(1 + \alpha_0 \mathrm{Bi}_2') + \mathrm{Bi}_2' \operatorname{ch} \alpha_0}, \quad b_1 = \frac{\mathrm{Bi}_1'(1 + \mathrm{Bi}_2')}{\mathrm{Bi}_1'(1 + \alpha_0 \mathrm{Bi}_2') + \mathrm{Bi}_2' \operatorname{ch} \alpha_0}, \tag{22}$$

$$\tilde{k} = \frac{c_1 \lambda_{gr}}{R_0}, \quad \alpha_0 = \ln\left[\frac{H}{R_0} + \sqrt{\left(\frac{H}{R_0}\right)^2 - 1}\right].$$

Equation (22) is similar to Shukhov's equation, differing from it only in that the temperature of the external medium T_e enters with a correction for heat transfer in the ground, and instead of the heat-transfer coefficient there appears the quantity k taking account of the thermal interaction of the pipeline with the surrounding medium.

The analytic relations (11) and (19) were compared with experimental data obtained at gas mains of the Middle Asia Center near the Khiva compressor station by B. L. Krivoshein, V. A. Trokhin, and A. V. Petrov and near the Urgench compressor station at Bukhara-Ural by B. L. Krivoshein, M. F. Sverdlov, and V. V. Spiridonov. The results of the comparison are shown in Fig. 1a,b. On these figures the numbers to the right of the chosen points (points of measurement) are the experimental results. The numbers to the left of the points are calculated. The upper values correspond to boundary conditions of the first kind, and the lower values to boundary conditions of the third kind. The calculated temperatures are in satisfactory agreement with experiment. The divergences from experiment for boundary conditions of the first kind average from 3-7% for the Bukhara-Ural gas main, and 5-10% for the Middle Asia Center gas main. For boundary conditions of the third kind the differences are no more than 10%. The differences arise from the fact that under actual conditions the thermal conductivity of the ground is different in different directions even when the ground is homogeneous. Close to the pipe the ground is dry and its thermal conductivity is lower than in regions far from the pipe. In addition, the ground is not homogeneous, and this affects the experimental results. On the whole it should be noted that the working formulas (11) and (16) are quite accurate enough for engineering applications.

NOTATION

T_{gr} is the temperature of the ground; T_e is the temperature of the external medium; T_F is the temperature of the fluid flowing in the cylindrical channel; T_{nl} is the temperature of the neutral layer; λ_{gr} is the thermal conductivity of the ground; α_1 is the coefficient of heat transfer from the fluid to the channel wall; α_e is the coefficient of heat transfer from the ground to the external medium; R_0 is the radius of the channel; H is the distance from the surface of the ground to the channel axis; H_0 is the depth of the neutral layer of the ground; λ_F is the thermal conductivity of the fluid; G is the mass flow rate of the fluid; c_p is the specific heat of the fluid; x, y, z, α, β are coördinates.

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